Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 4, pp. 601–627, 1994

0021–8928/94 $24.00+0.00

0021-8928(94)00087-5

THE "PODMODELI" PROGRAM. GAS DYNAMICS†

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(*Received* 15 *November* 1993)

Initial ideas are presented and the characteristic features of the implementation of the "PODMODELI" ("submodels") program, which has been written during recent years [1], are described. At the foundation of this program lies a deliberation concerning the exhaustion of all the possibilities of an exact simplification of "large" mathematical models by means of the full use of the symmetry properties built into them. Such a simplification is achieved by changing to submodels which describe classes of exact particular solutions and lead to a reduction in the dimensions of the problems making them easier to analyse. Although, in its full scope, this program is obviously not exhaustible, nevertheless, as regards each fixed "large model" which arises in the mechanics of a continuous medium and, in general, in mathematical physics, it is specific and fully realizable in practice and serves for the organization and supplementation of a databank on mathematical models of natural processes.

In Section 1 we describe the conceptual basis of the "PODMODELI" program, the principal idea, aim, content, theoretical foundations and overall scientific significance. The most important operational algorithms which are used in its execution are pointed out. Section 2 contains initial information concerning "large models" of gas dynamics and their overall symmetry. In Section 3, we present a general description of the algorithm for group classification and the result of applying it to the equations of gas dynamics. The most important concepts associated with the algorithm for constructing the optimal systems of sub-algebras in the case of finite-dimensional Lie algebras are collected together in Section 4. As an example, we present the result of the application of this algorithm to the Lie algebra of the L_{11} group which is assumed by the equations of gas dynamics in the case of an equation of state of general form. Section 5 describes all (apart from similitude) the invariant submodels of rank three for "large models" of gas dynamics with a common equation of state. The concluding Table 6, which is the normalized optimal system of sub-algebras in the case of the L_{11} Lie algebra, is presented in the appendix.

1. THE IDEA OF THE SUBMODELS PROGRAM

In spite of the fact that symmetry properties have always attracted the attention of investigators, there has as yet been no complete concept of the systematic use of these properties in the mechanics of continuous media. It is noteworthy that, in modern theoretical physics, group-theoretic methods play a fundamental role in the investigation of the structure of matter (the microcosm) and of the universe (the macrocosm). There is a glaring omission regarding this matter in the mechanics of continuous media which has arisen for various reasons and quite obviously needs to be made good.

†*Prikl. Mat. Mekh.* Vol. 58, No. 4, pp. 30–55, 1994.

Below, we present a brief description of one of the areas of investigation to partially rectify this omission in the form of a certain program which has been named the PODMODELI ("submodels") program.

A system of relationships, consisting of differential equations and supplementary relations which describes the characteristics of the distribution of the physical quantities in space and their evolution with time is referred to by the term "large model". The symmetry property of a "large model" reflects the fact that the laws of nature built into it are independent of measurement (reference) systems and expresses the invariance of the values of the principal quantities with respect to certain spatial transformations. These transformations form a group.

Following Lie, we shall say that a system of differential equations E *admits* of a group G of transformations of all the quantities (both the independent and dependent variables) participating in E if the system E remains invariant under all transformations belonging to the group G. In the case of the equations of mathematical physics and the mechanics of continuous media, the classical groups of Euclid, Galileo, Poincaré and so on as well as their subgroups and extensions, which are permitted by them, are characteristic. Here, the symmetry of a given model can be extended in the case of particular forms of the supplementary links (in the case of certain relationships).

A fundamental property of a group G which is allowed by a system E lies in the fact that *the group G acts on the set of all solutions of E.* This is also true for any subgroup $H \subset G$.

Each subgroup $H \subset G$. has invariants which are finite and/or differentiable. The establishment of additional relationships between the invariants of a subgroup H picks out a class of exact particular solutions, the so-called *H-solutions*, from the set of all solutions of E. These solutions are expressed in terms of new required functions (invariants) which satisfy the system of differential equations derived from E referred to as the *factor system E/H.* The factor system is usually simpler compared with the initial system E and, in particular, this is due to the fact that E/H contains a smaller number of independent variables. The factor system E/H is therefore referred to as a submodel of the initial "large model" E. The number of independent variables in E/H is called the rank of the submodel. In the standard case of a four-dimensional space of events, in which system E is defined, the rank of a submodel can take the values 3, 2, 1, 0.

The majority of known exact submodels in the form of systems of equations of reduced dimensions such as one-dimensional, two-dimensional, plane-parallel, axially symmetric, spiral, stationary, conical and self-similar systems describe *invariant* H-solutions. This list is obviously incomplete, but the question of how to complete it has remained unclear until recently.

The idea behind the introduction of the PODMODELI program is that, by means of the systematic use of the symmetry property of a given "large model" E, one can set up a *comprehensive inventory* (databank) of the exact submodels which are generated by model E in the form of classes of H-solutions which are described by the factor systems E/H.

The realization of the PODMODELI program must lead to the maximal recovery of the possibilities which are built into the symmetry properties of "large models" in the mechanics of continuous media and thereby to the enrichment of the theory of the phenomena which are being described by these models. This enables one to provide a solution to many new specific problems, to reveal additional peculiarities of the processes being described, to obtain a broad set of tests for the testing of numerical methods and to broaden the field of topics which are used in the system for the training of specialists.

Group theory, the theory of Lie algebras and the group analysis of differential equations form the fundamental basis for the realization of the PODMODELI program. The specific implementation involves the use of fairly well tested algorithms. Among these algorithms, the following are the most important.

The group classification algorithm acts in case where the given system of differential equations E contains an *arbitrary element* A (such a system is denoted by $E(A)$) in the form of undetermined parameters and functions which are supplementary in E on account of any additional relationships between the basic quantities. This algorithm is required by virtue of the fact that the specialization of the arbitrary element A can lead to an extension of the allowed group. This algorithm is not only applied to "large models" but also to their subgroups of various ranks. The algorithm for calculating the basic Lie group which is permitted by system E is a special case of the above-mentioned algorithm. A schematic description of the group classification algorithm is given below in Section 3.

The algorithm for constructing the optimal system of subgroups of G (or sub-algebras of the corresponding Lie algebra L) permitted by system E is required due to the fact that submodels constructed with reference to the different subgroups H_1 and H_2 of group G may turn out to be *similar* (equivalent). In this case, all of the H_1-solutions are obtained from the H_1 solutions by a certain transformation of the basic quantities (by a change of variables). This occurs when the subgroups H_1 and H_2 are conjugated (similar) in group G with respect to the *internal automorphisms* of this group. It is therefore important that different submodels should only be obtained with respect to different classes of conjugated subgroups. The set of representatives of such classes is called the *optimal system* of subgroups and is denoted by the symbol ΘG (the optimal system of sub-algebras ΘL respectively). Here, it is also useful to remark that the factor system E/H always admits of the normalizer of subgroup H in G, that is, one has "a priori" knowledge of the group which is permitted by a submodel. The main points associated with the algorithm for constructing optimal systems of sub-algebras of finite-dimensional Lie algebras are discussed in Section 4.

An algorithm for reducing a system of differential equations to an involution is employed in the realization of the PODMODELI program. It acts every time overdefined systems of equations arise and, in particular, when calculating the basic permitted group, in the group classification and in the analysis of partially invariant solutions. This algorithm is not considered in the present paper. Further details can be found in [6], for example.

The PODMODELI program anticipates the performance of a preliminary analysis of the qualitative behaviour of the solutions of the resulting factor systems E/H. This attending to the submodel (or, as we say, "dressing") may include a description of the type of system E/H and the formulation of the principal boundary-value problems, the study of the structure of the set of trajectories, the characteristics, strong discontinuities, etc.

It is pertinent to note that the process of realizing the PODMODELI program does not reduce to the purely applied use of a ready piece of apparatus for mathematical analysis. In fact, many new questions and unsolved problems arise during this process, and the treatment of these is of general theoretical importance. In particular, this refers to the theory of integration of overdefined systems of differential equations with supplementary structure, to the general theory of invariants including the differential invariants of Lie groups of transformations and to the purely algebraic theory of subgroup (sub-algebraic) structures of Lie groups (algebras)

On the whole, the PODMODELI program is completely realistic but extremely laborious, and its use requires a great deal of collective work. It is to be hoped that the PODMODELI program will attract the attention and receive the support of the scientific community of specialists in the fields of mathematical physics and the mechanics of continuous media.

2. "LARGE MODELS" OF GAS DYNAMICS

A "large model" of gas dynamics was chosen for the start of the work using the PODMODELI program for a number of reasons. A great deal of scientific experience has been accumulated in gas dynamics on various kinds of work with exact solutions. Because of the richness of its content, it provides an excellent example for the testing of all the features in the implementation of the PODMODELI program. There is a certain store for it [2]. Although there is a set of papers which deal with the use of various different manifestations of the symmetry properties of the equations of gas dynamics, the possibilities which are hidden in this property are still in need of systematic study.

We are concerned with the description of the motion of a gas as a two-parameter continuous medium when there are no dissipation and external force fields. The initial "large model" is the system of differential equations (in dimensionless variables)

$$\rho D\mathbf{u} + \nabla p = 0, \quad D\rho + \rho \operatorname{div} \mathbf{u} = 0, \quad Dp + A \operatorname{div} \mathbf{u} = 0 \tag{2.1}$$

where $D = \partial_t + \mathbf{u} \cdot \nabla$, $\nabla = (\partial_x, \partial_y, \partial_z)$, $\mathbf{u} = (u, \upsilon, w)$ is the velocity vector, ρ is the density and p is the pressure. These quantities are functions of the time, t, and the coordinates $\mathbf{x} = (x, y, z)$. It is assumed that the state function $A = A(p, \rho)$ is given. The physical meaning of the function A is defined by its expression $A = \rho c^2$, where c is the velocity of sound.

It should be noted that the last equation in (2.1) is conventionally written with the entropy S in the form $DS = 0$ and linked with the equation of state $p = F(\rho, S)$. The two forms of writing the equation are equivalent, subject to the condition that $F_S \neq 0$. The assumption that $F_S = 0$ leads to the treatment of a class of isentropic motions of the gas (the unfortunate term "barotropic gas" is frequently used) and must be the object of an independent study.

The nine-dimensional space $R^9(t, \mathbf{x}, \mathbf{u}, p, \rho)$ is the *basic space* in the case of system (2.1). It is known [2] that, with functions $A(p, \rho)$ of a general form, system (2.1) permits an 11-parameter Lie group G_{11} of transformations of the space R^9 which is generated by *transfer* (translational) transformations with respect to the variables t, $\mathbf{x}$: $t \to t + t_0$, $\mathbf{x} \to \mathbf{x} + \mathbf{x}_0$, *Galilean translational* transformations in the direction of the axes $\mathbf{x}: \mathbf{x} \to \mathbf{x} + \mathbf{u}_0 t$, $\mathbf{u} \to \mathbf{u} + \mathbf{u}_0$, *rotational* transformations in the subspace $R^6(\mathbf{x}, \mathbf{u})$ $:\to S_0\mathbf{x}$, $\mathbf{u} \to S_0\mathbf{u}$ and homogeneous *extensions* (homotheties) of the subspace $R^4(t, \mathbf{x})$: $t \to k_0 t$, $\mathbf{x} \to k_0 \mathbf{x}$. Here, t_0, $\mathbf{x}_0$, $\mathbf{u}_0$, k_0 are arbitrary parameters and S_0 is an orthogonal (3×3)-matrix with a determinant $S_0 = 1$. Algebraically, group G_{11} is a Galilean G_{10} group expanded by means of a one-dimensional group of homotheties. Moreover, system (2.1) permits two discrete transformations (*involutions*) $I_1 : (\mathbf{x}, \mathbf{u}) \to (-\mathbf{x}, -\mathbf{u})$, $I_2 : (t, \mathbf{u}) \to (-t, -\mathbf{u})$ which correspond to a change in the orientation (I_1) of the subspace $R^6(\mathbf{x}, \mathbf{u})$ and time inversion (I_2).

The Lie algebra L_{11}, corresponding to G_{11}, of the operators defined in the space R^9 participates in the specific calculations. The operators X_i $(i = 1, \ldots, 11)$, which are respectively selected by the transformations listed above

$$\begin{aligned}
&X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_z \\
&X_4 = t\partial_x + \partial_u, \quad X_5 = t\partial_y + \partial_\upsilon, \quad X_6 = t\partial_z + \partial_w \\
&X_7 = y\partial_z - z\partial_y + \upsilon\partial_w - w\partial_\upsilon, \quad X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w \\
&X_9 = x\partial_y - y\partial_x + u\partial_\upsilon - \upsilon\partial_u, \quad X_{10} = \partial_t \\
&X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z
\end{aligned} \tag{2.2}$$

form a basis in L_{11}.

From the point of view of group analysis, the function $A(p, \rho)$ in system (2.1) is treated as an arbitrary element with respect to which a group classification of this system has to be carried out within the framework of the PODMODELI program. This means that one has to find all such forms of the function $A(p, \rho)$ (apart from equivalence) with which an expansion of the group G_{11} occurs. This classification is basically known [2]. Some of its details are presented as an example in Section 3. Here, only the refined results are communicated.

The operators

$$\begin{aligned}
&Y_1 = t\partial_t - u\partial_u - \upsilon\partial_\upsilon - w\partial_w + 2\rho\partial_\rho, \quad Y_2 = \rho\partial_\rho + p\partial_p, \quad Y_3 = \partial_p \\
&Y_4 = t^2\partial_t + tx\partial_x + ty\partial_y + tz\partial_z + (x - tu)\partial_u + (y - y\upsilon)\partial_\upsilon + (z - tw)\partial_w - 3t\rho\partial_\rho - 5tp\partial_p
\end{aligned} \tag{2.3}$$

which are additional to (2.2) participate in expansions of the L_{11} Lie algebra.

A summary of all cases of expansion is given in Table 1. The numbers of expansions are given in column N, the forms of this function which produce an expansion of L_{11} are shown in column A while column p contains the corresponding forms for the conventional representation of the equation of state $p = F(\rho, S)$. The numbers in column k denote the dimensions of the expanded L_k algebras. The operators in the notation of (2.3) which augment the L_{11} basis (2.2) up to a basis of an L_k Lie algebra are written out in column Y. The arbitrary entropy function is denoted by the symbol э that is, $\text{э} = \text{э}(S)$, and f is an arbitrary function of the above-mentioned arguments which is different in the different cells of the table.

Among the "large models" of gas dynamics listed in Table 1, widely known models are encountered which have been the subject of numerous investigations. Apart from the common model $N = 1$, we may mention the model $N = 6$ of a polytropic gas and its special case $N = 7$. The model $N = 13$ contains a general model of an ideal gas. The case $N = 4$ is distinguished by the special properties of this model in the case of steady gas flows [5]. The exotic model $N = 13$ describes a gas with zero velocity of sound. Regardless of its physical meaning, it is distinguished by the fact that the group permitted by it is infinitely dimensional on account of the allowed operator $Y_\varphi = \rho\varphi'(p)\partial_\rho + \varphi(p)\partial_p$ with an arbitrary function $\varphi(p)$. Moreover, some of the "large models" which have been found such as $N = 2$, 5, 9–12, for example, have not been studied in any detail.

As a result, it is found that, in the implementation of the PODMODELI program as applied to gas dynamics, 13 "large models" have to be considered. This work was begun in the case of the model $N = 1$ which is discussed in this paper.

3. GROUP CLASSIFICATION

Let a certain system of differential equations, defined in a *basis space* $R^{n+m}(z)$, where $z = (x, y)$ and $x = (x^1, \ldots, x^n)$ is a set of independent variables while $y = (y^1, \ldots, y^m)$ is the set of required functions, contain an arbitrary element $A = (A^1, \ldots, A^p)$, the components of which, A^i, may be functions of z and obey certain additional conditions. This system is denoted by $E(A)$ and the additional conditions by $\Omega(A)$.

The problem involves finding the group of transformations of the basis space $R^{n+m}(z)$, which are permitted by the system $E(A)$ in the case of an arbitrary element of general form which solely satisfies the conditions $\Omega(A)$ for all possible specializations of A which lead to expansions of the permitted groups.

Table 1

N	A	p	k	Y
1	$f(p, \rho)$	$f(\rho, \text{э})$	11	–
2	$pf(p\rho^{-\gamma}),\ \gamma \neq 0,1$	$\rho^\gamma f(\text{э}\rho)$	12	$(\gamma-1)Y_1 - 2\gamma Y_2$
3	$pf(p/\rho)$	$\rho f(\text{э}\rho)$	12	Y_2
4	$f(p)$	$f(\text{э}\rho)$	12	Y_1
5	$pf(\rho)$	$\text{э} f(\rho)$	12	$Y_1 - 2Y_2$
6	$\gamma p,\ \gamma \neq 0, 5/3$	$\text{э}\rho^\gamma$	13	$Y_1,\ Y_2$
7	$(5/3)p$	$\text{э}\rho^{5/3}$	14	$Y_1,\ Y_2,\ Y_4$
8	$f(\rho e^{-p})$	$\ln\rho + f(\text{э}\rho)$	12	$Y_1 + 2Y_3$
9	$f(\rho)$	$f(\rho) + \text{э}$	13	Y_3
10	$\gamma\rho^\gamma,\ \gamma \neq 0,1$	$\rho^\gamma + \text{э}$	13	$(\gamma-1)Y_1 - 2\gamma Y_2,\ Y_3$
11	ρ	$\rho + \text{э}$	13	$Y_2,\ Y_3$
12	1	$\ln\rho + \text{э}$	13	$Y_1,\ Y_3$
13	0	э	∞	$Y_1,\ Y_\varphi$

For simplicity in the discussion below, this problem is considered for systems $E(A)$ of the first order and additional conditions $\Omega(A)$ which also contain derivatives of A which are no higher than of the first order. In this case, the initial equations can be written in the form

$$E(A):\ \ E(z_1, A(z)) = 0; \quad \Omega(A):\ \ \Omega(z, A_1(z)) = 0 \tag{3.1}$$

where the combined sets of all the main quantities and their first derivatives are denoted by the subscript 1. Together with Eqs (3.1), we introduce an auxiliary system of equations

$$E(a):\ \ E(z_1, a) = 0 \tag{3.2}$$

with the arbitrary quantities $a = (a^1, \ldots, a^p)$, which are substituted into $E(A)$ instead of the corresponding functions $A(z)$. The algorithm for solving the problem which has been posed can be conveniently subdivided into four stages.

The conditions for the invariance of the equation $E(a)$ with respect to single parameter groups of transformations of the space $R^{n+m}(z) \times R^p(a)$ are formed during the first stage. The operators of these groups are sought in the form

$$X^a = \xi^z \partial_z + \xi^a \partial_a = \xi^x \partial_x + \xi^y \partial_y + \xi^a \partial_a \tag{3.3}$$

where the coordinates ξ^z depend solely on z, and the coordinates ξ^a may be functions of the variables z and a (the possible generalization when ξ^z is allowed to be dependent on a is not considered here). A scalar product, that is, the sum over all values of a repeated index $\xi^x \partial_x = \xi^{x^1} \partial_{x^1} + \xi^{x^2} \partial_{x^2} + \ldots + \xi^{x^n} \partial_{x^n}$ and so on, is denoted by a dot in expression (3.3) and below. The condition for the invariance of $E(a)$ has the form (see [2], for example)

$$X_1^a E(z_1, a)|_{E(a)} = 0 \tag{3.4}$$

where X_1^a is the first continuation of the operator X^a in the arbitrary $y_x = (y^1_{x^1}, \ldots, y^m_{x^n})$ and is written as

$$X_1^a = X^a + \xi^{y_x} \partial_{y_x} \tag{3.5}$$

The coordinates of the extended operator are calculated using standard formulae (a sum over $x' = x^1, \ldots, x^n$)

$$\xi^{y_x} = D_x \xi^y - y_{x'} D_x \xi^{x'}, \quad D_x = \partial_x + y_x \partial_y \tag{3.6}$$

for each $x = x^1, \ldots, x^n$ and each $y = y^1, \ldots, y^m$. A system of equations $DE(a)$ in the required coordinates of the operator (3.3) is obtained from (3.4). This consists of two subsystems and, in fact, $DE(a) = DE_0 \cup DE_a$. The equations DE_0 do not contain the variables a or ξ^a while the equations DE_a contain a and relationships of the form

$$m_a \xi^a = l_z \xi^z \tag{3.7}$$

in which l_z are linear (inhomogeneous) differential operators with respect to the variables z, which act in ξ^z and m_a are certain functions. Moreover, l_z and m_a depend on z and a.

The group of *equivalence transformations* of equation $E(a)$ which also preserve the additional conditions $\Omega(a)$, which the vector $a = A(z)$, must satisfy and are written in the form

$$\Omega(a):\ \ \Omega(z, a_1) = 0 \tag{3.8}$$

is calculated during the second stage.

The invariance condition $\Omega(a)$ has the form

$$Y_1^a \Omega(z,a_1)|_{\Omega(a)} = 0 \tag{3.9}$$

in which the extension Y_1^a of the X^a in the arbitrary $a_1 = (a_z^i)$ is used. This extension has to be calculated using the formulae

$$\begin{aligned} Y_1^a &= X^a + \xi^{a_z}\partial_{a_z} \\ \xi^{a_z} &= D_z^+\xi^a + a_{z'}D_z^+\xi^{z'}, \quad D_z^+ = \partial_z + a_z\cdot\partial_a \end{aligned} \tag{3.10}$$

The invariance condition (3.9) yields an additional subsystem of governing equations $D^+\Omega_a$. Furthermore, the fact that the coordinates ξ^z are independent of the variables a is taken into account. As a result, one obtains the complete system of governing equations

$$DE_0 \cup DE_a \cup D^+\Omega_a \tag{3.11}$$

for the coordinates of the operators

$$X^a = \xi^z(z)\cdot\partial_z + \xi^a(z,a)\cdot\partial_a$$

which generate the group of equivalence transformations of the system of equations $E(a)\cup\Omega(a)$.

According to general theory [2], the *kernel of the principal groups* of equation $E(A)$, that is, the group which is permitted by these equations for *any* function $a = A(z)$ which satisfies the additional condition $\Omega(A) = 0$, is contained in the group of equivalence transformations. The kernel is therefore defined by Eqs (3.7), in which it is necessary to put $\xi^a = a_z\xi^z$ and, in the resulting equality, to pass into a manifold specified by the equations $\Omega(a)$.

It is pertinent to note that the kernel of the principal groups is always an *invariant subgroup* of the group of equivalence transformations.

The third stage involves the construction and solution of the governing equations of the coordinates of the operators X^A which are permitted by the system $E(A)\cup\Omega(A)$. A part of these equations, DE_0, in fact, has already been obtained during the first stage. The remaining parts are simply obtained from the subsystem $DE_a \cup D^+\Omega_a$ by substituting $a = A(z)$ and replacing the coordinates ξ^a in relationships (3.7) by the expressions $\xi^A = A_z\xi^z$. As a result of this, they transform into the equations

$$D^+\Omega_A:\ m_A\cdot A_z\cdot\xi^z = l_z\cdot\xi^z \tag{3.12}$$

The system of governing equations obtained by this means

$$DE_0 \cup DE_A \cup D^+\Omega_A \tag{3.13}$$

is a system of differential equations in the required coordinates ξ^z of the operator X^A.

Generally speaking, the problem of constructing a general solution of the system of equations (3.13) is non-trivial. The search for this solution is facilitated by the existence in the case of system (3.13) of a number of special properties. Firstly, the system is linear and homogeneous and, as a rule, is strongly overdefined. Secondly, system (3.13) is invariant with respect to the group of equivalence transformations constructed during the second stage. Furthermore, it is known beforehand that the commutator of any two solutions of this system is again its solution and that the operators of the kernel of the principal groups found during the second stage yield its particular solutions.

One usually proceeds in the following manner: initially, if possible, one finds the general form of the coordinates ξ^z, which satisfy the equations DE_0. By virtue of linearity, these coordinates are represented in the form of linear combinations of certain functions $\varphi(z)$ which are arbitrary up to now (and which may also be arbitrary constants). The representations obtained are substituted into the remaining equations $DE_A \cup D^+\Omega_A$ which thereby become equations in the functions φ which form an overdefined system $DE_A(\varphi)$. The system $DE_A(\varphi)$ is subsequently brought into *involution*, that is, it is represented (as a rule, by means of an extension) in such a form that the width of the general solution is explicitly determined. The process of bringing into involution is associated with the formation of compatibility conditions and, generally speaking, can branch depending on the actual form of the function $A(z)$ occurring in the system $DE_A(\varphi)$. The general solution of the system of governing equations with each fixed function $A(z)$ yields all the operators X^A which generate the widest (principal) group which is permitted by the system $E(A)$.

The group classification of the systems $E(A)$ is carried out during the fourth stage. This classification is associated with the above-mentioned branching during the process of involution. Each branch gives rise to the need to take account of the fact as to whether certain differential expressions $d(z, A)$ in the function A are equal or not equal to zero. The alternative equations

$$d(z, A) = 0 \tag{3.14}$$

which arise in this case are *classifying* equations: actually, they pick out the possible expansions of the permitted group.

According to the construction, the group of equivalence transformations acts on a set of solutions of the classifying equations (3.14). This enables one to pick out their characteristic solutions which have the simplest analytic form.

Finally, for each solution of Eqs (3.14) which has been picked out, a general solution of the system $DE_A(\varphi)$ is found. This also yields the operators X^A which generate the principal group permitted by the system $E(A)$ for the corresponding form of the "arbitrary element" $A(z)$. The final result of the group classification is represented in the form of a table containing all the cases of the expansion of the kernel of the principal groups.

Actually, the group classification of "large models" $E(A)$ of gas dynamics (2.1) represented in Table 1 was obtained by means of this algorithm. Here, the "arbitrary element" $a = A(p, \rho)$ satisfies the additional conditions

$$\Omega(A): \; A_t = A_{\mathbf{x}} = A_{\mathbf{u}} = 0 \tag{3.15}$$

In the first stage, the required operators are written in the form

$$X^a = \xi^t\partial_t + \xi^{\mathbf{x}}\cdot\partial_{\mathbf{x}} + \xi^{\mathbf{u}}\cdot\partial_{\mathbf{u}} + \xi^p\partial_p + \xi^\rho\partial_\rho + \xi^a\partial_a$$

where $\xi^{\mathbf{x}}\cdot\partial_{\mathbf{x}} = \xi^x\partial_x + \xi^y\partial_y + \xi^z\partial_z$ and so on. Here and henceforth, derivatives with respect to corresponding variables are denoted using subscripts.

The well-known criterion [3] shows that system (2.1) is "x-autonomous" for any A, that is, the coordinates ξ^t, $\xi^{\mathbf{x}}$ may depend solely on the variables t, $\mathbf{x}$. When account is taken of this and subject to the condition that $a \neq 0$ the subsystem DE_0 reduces to the following. The coordinate ξ^t is independent of x and is solely a function of t. The coordinates $\xi^{\mathbf{x}}$ satisfy the equations

$$\xi^x_x = \xi^y_y = \xi^z_z; \quad \xi^x_y + \xi^y_x = 0, \quad \xi^y_z + \xi^z_y = 0, \quad \xi^z_x + \xi^x_z = 0 \tag{3.16}$$

All of the second-order partial derivatives of these coordinates with respect to the variables t and $\mathbf{x}$ are equal to zero apart from the mixed derivatives $\xi^x_{xt} = \xi^y_{yt} = \xi^z_{zt}$ which satisfy the equation

$$2\xi^x_{xt} = \xi^t_{tt} \tag{3.17}$$

The coordinates $\xi^{\mathbf{u}}$ and ξ^p have the actual expressions

$$\begin{aligned}
\xi^u &= \xi^x_t + u(\xi^x_x - \xi^t_t) + v\xi^x_y + w\xi^x_z \\
\xi^v &= \xi^y_t + u\xi^y_x + v(\xi^x_x - \xi^t_t) + w\xi^y_z \\
\xi^w &= \xi^z_t + u\xi^z_x + v\xi^z_y + w(\xi^x_x - \xi^t_t) \\
\xi^\rho &= \rho(\xi^p_p + 2\xi^t_t - 2\xi^x_x)
\end{aligned} \tag{3.18}$$

Finally, the coordinate ξ^p does not depend on the variables $\mathbf{x}, \mathbf{u}$ and ρ, that is, it is solely a function of t and p and satisfies the equations

$$\xi^p_{pp} = 0, \quad \xi^p_{pt} + 5\xi^x_{xt} = 0 \tag{3.19}$$

Equations (3.16)–(3.19) also constitute the subsystem DE_0. The subsystem DE_a reduces to the equations

$$\xi^p_t + 3a\xi^x_{xt} = 0, \quad \xi^a = a\xi^p_p \tag{3.20}$$

The second stage (the calculation of the group of equivalence transformations) reduces in this case to taking account of the fact that the function ξ^p and ξ^x are independent of a, which yields $\xi^p_t = 0$ and $\xi^x_{xt} = 0$. All of the second-order derivatives of the coordinates ξ^t, $\xi^{\mathbf{x}}$ are therefore equal to zero. As far as the additional conditions $\Omega(a)$ are concerned, which have the form $a_t = a_{\mathbf{x}} = a_{\mathbf{u}} = 0$ here, the condition for their invariance, which is readily verified, is identically satisfied. The construction of the general solution of the resulting system of governing equations which already occurs in involution is carried out automatically. As a result, one obtains a 14-parameter group of equivalence transformations generated by the operators X^A with the coordinates

$$\begin{aligned}
\xi^t &= C_{10} + (C_{11} - R_1)t \\
\xi^x &= C_1 + C_4 t + C_{11}x - C_9 y + C_8 z \\
\xi^y &= C_2 + C_5 t + C_9 x + C_{11}y - C_7 z \\
\xi^z &= C_3 + C_6 t - C_8 x + C_7 y + C_{11}z \\
\xi^u &= C_4 + R_1 u - C_9 \upsilon + C_8 w \\
\xi^\upsilon &= C_5 + R_1 \upsilon + C_9 u - C_7 w \\
\xi^w &= C_6 + R_1 w - C_8 u + C_7 \upsilon \\
\xi^\rho &= (2R_1 + R_3)\rho; \quad \xi^p = R_2 + R_3 p; \quad \xi^a = R_3 a
\end{aligned} \tag{3.21}$$

where C_i, R_j are arbitrary constants.

The kernel of the principal groups is distinguished by the expression $\xi^a = \xi^p a_p + \xi^\rho a_\rho$ which reduces to the equation

$$(R_2 + R_3 p)a_p + (-2R_1 + R_3)a_\rho = 0$$

which, by virtue of the arbitrariness of the quantities a_p, a_ρ yields $R_1 = R_2 = R_3 = 0$. The kernel is therefore determined by the constants C_i in (3.21) and is identical with the group G_{11}, the operators of which are written out in (2.2). The factor group with respect to this kernel is generated by operators corresponding to the constants R_j

$$\begin{aligned}
X^a_1 &= -t\partial_t + u\partial_u + \upsilon\partial_\upsilon + w\partial_w + 2\rho\partial_\rho \\
X^a_2 &= \partial_p, \quad X^a_3 = p\partial_p + \rho\partial_\rho + a\partial_a
\end{aligned} \tag{3.22}$$

The equivalence transformations of the "arbitrary element" A which follow from here form a 3-parameter group which acts according to the formulae (a transformed quantity is denoted by a prime)

$$p' = \alpha_3 p + \alpha_2, \quad \rho' = \alpha_1\alpha_3\rho, \quad A' = \alpha_3 A \tag{3.23}$$

with arbitrary parameters α_j and $\alpha_1 > 0$ and $\alpha_3 > 0$.

In the third stage, it is assumed that $a = A(p, \rho)$ and $\xi^a = \xi^A = A_p\xi^p + A_\rho\xi^\rho$ in (3.20) which leads to the equations

$$\xi_t^p + 3A\xi_{xt}^x = 0, \quad A_p\xi^p + A_\rho\xi^\rho = A\xi_p^p \tag{3.24}$$

Together with (3.15)–(3.19), they also form a complete system of governing equations for the operators which are allowed by system (2.1).

In order to bring this system into involution, account is first taken of the fact that the quantity ξ_{xt}^x is a constant. Let $\xi_{xt}^x = B_0$. It then follows from the first equation of (3.19) that ξ^p is a linear function of p and one can put

$$\xi^p = \varphi(t) + \psi(t)p \tag{3.25}$$

Then, for ξ^ρ in (3.18), we obtain the expression

$$\xi^\rho = \rho(\psi + 2\xi_t^t - 2\xi_x^x) \tag{3.26}$$

The remaining equations of (3.19) and (3.24) reduce to the following (derivatives with respect to t are denoted by primes)

$$\psi' = -5B_0, \quad \varphi' = -(3A - 5p)B_0 \tag{3.27}$$

$$(\varphi + \psi p)A_p + (\psi + 2\xi_t^t - 2\xi_x^x)\rho A_\rho = \psi A \tag{3.28}$$

Differentiating (3.28) with respect to t and taking account of (3.27) and (3.17) we obtain the equation $(3AA_p + 3\rho A_\rho - 5A)B_0 = 0$. On the other hand, the equality $(3A_p - 5)B_0 = 0$ is obtained by differentiating the second equation of (3.27) with respect to p. As a result, one obtains the compatibility conditions for the governing equations

$$(3A_p - 5)B_0 = 0, \quad A_\rho B_0 = 0 \tag{3.29}$$

which give rise to branching of the solution process: either $B_0 = 0$ or $B_0 \neq 0$.

If $B_0 = 0$, it follows from (3.27) that $\varphi' = \psi' = 0$ and, moreover, all second derivatives with respect to the coordinates ξ^t, ξ^x are equal to zero. In this case ξ_x^x and ξ_t^t are constants. Let $\xi_x^x - \xi_t^t = \omega$. Then, Eq. (3.28), rewritten in the form

$$\varphi A_p + \psi(pA_p + \rho A_\rho - A) + 2\omega\rho A_\rho = 0 \tag{3.30}$$

is unique and the group constants φ, ψ, ω must satisfy it. By virtue of (3.30), not more than two of them can be free. It means that, in this case, the kernel (2.2) can only be expanded in not more than two operators. The subsequent analysis of the expansions of the kernel of the principal groups is solely associated with Eq. (3.30) and the use of the equivalence transforms (3.23). This can be carried out using various specific methods (see [2], for example). As a result, all cases of expansion listed in Table 1, apart from the case $N = 7$, are obtained.

If $B_0 \neq 0$, it follows from (3.29) that $A_{\rho=0}$ and $3A_p = 5$ whereupon $A = 5p/3 + C$, where $C = \text{const}$. The constant C can be made equal to zero by means of the equivalence transformations (3.23). This yields the classification case $A = 5p/3$ which corresponds to $N = 7$ in Table 1. For this function A, $\varphi = 0$, is obtained from (3.28) and $\psi = -5B_0t + R_2$ from (3.27). Moreover, here we shall have $\xi_x^x = B_0t + C_{11}$, and Eq. (3.17) can be integrated with respect to t in the form $\xi^t = B_0t^2 + (C_{11} - R_1)t + C_{10}$. The kernel is therefore expanded by the operators Y_1, Y_2 and Y_4 by means of the constants R_1, R_2 and B_0, respectively. The group classification of system (2.1) with respect to an "arbitrary element" $A(p, \rho)$ is concluded.

4. OPTIMAL SYSTEMS OF SUB-ALGEBRAS

In order to calculate all of the submodels of gas dynamics which can be obtained on the basis of the symmetry properties mentioned in Section 2, a classification of the subgroups of the allowed groups, apart from conjugation, has to be carried out for each "large model" from Table 1. Here, the solution of this problem is demonstrated using the example of the G_{11} group, which is allowed by Eqs (2.1) with an equation of state of general form. Actually, an equivalent problem on the classification of the sub-algebras of L_{11} Lie algebra with a basis (2.2) is solved. Here, we make use of certain general corollaries from the theory of Lie algebras. The reader may familiarize himself with these using various books (see [7], for example).

In the general case, a finite-dimensional Lie algebra L of dimensions n over a field of real numbers with a basis $X_1, \dots, X_n$ and a table of commutators $[X_i, X_j]=C_{ij}^k X_k$ $(i, j=1, \dots, n)$ is the initial object. The *augmented group* A of internal automorphisms of the Lie algebra L is calculated using this table by integrating a system of Lie differential equations.

Group A acts on a set of sub-algebras $K \subset L$ by virtue of which this set is separated into classes of coupled (similar) sub-algebras. The set of representatives of these classes (one each from each class) is called the *optimal system* of sub-algebras and is denoted by the symbol $\Theta_A L$ (the index "A" will sometimes be omitted for brevity).

A considerable amount of work including both purely algebraic operations for the separated classes of Lie algebras and their sub-algebras as well as operations with an applied trend in the case of the actual low-dimension Lie algebras (see [8–12], for example) is involved in the solution of the problem of constructing $\Theta_A L$.

The construction presented below is based on the use of certain special constructions [4]. The use of a composite series of ideals and the expansion of a Lie algebra $L=J\oplus N$ into a half line sum of a characteristic ideal J and a sub-algebra N with a corresponding expansion of the group $A=A_J A_N$ belong to such special constructions.

A two-stage algorithm is operating in which the optimal system $\Theta_{A_N} N=\{N_p \mid p\in P\}$ is constructed and the stabilizers $A_p \subset A$ of the sub-algebras N_p are determined during the first stage and the optimal systems $\Theta_{A_p}(J+N_p)=\{K_{p,q} \mid q\in Q_p\}$ are determined during the second stage. The required optimal system is given by the union

$$\Theta_A L=\{K_{p,q} \mid p\in P, q\in Q_p\}$$

The calculations are carried out in a coordinate representation $H_\alpha = x_\alpha^i X_i$ of the bases $H_\alpha(\alpha=1, \dots, r)$ of the required sub-algebras in the form of matrices $\xi = \| x_k^i \|$ taking account of the action on these matrices of the group B of transformations of the basis. The construction of $\Theta_A L$ thereby reduces to calculating the optimal system of matrices (which satisfy the equations of the sub-algebra) $\Theta_G\{\xi\}$ with respect to the action of the group $G=AB$.

The additional requirement that the required optimal system should be normalized means that, together with any sub-algebra, its normalizer $\mathrm{Nor}_L K \in \Theta_A L$ also. It is known that this quite rigorous requirement can always be satisfied [9]. The greatest possible number of *null coordinates* in the matrix representation of the bases of the sub-algebras has to be obtained. Coordinates which remain undetermined must be the invariants of the participating subgroups of group G.

The result of the calculation of the optimal system of sub-algebras is usually represented in the form of a table of sub-algebras with an indication of their bases. A more obvious (and convenient in practice) graphical representation is also possible in the case of a normalized optimal system ΘL. This is based on the fact that each sub-algebra belongs to a *branch of ideals* $K_1 \to K_2 \to \dots \to K_m$, in which $K_{i+1}=\mathrm{Nor}K_i$ $(i=1, \dots, m-1)$, and the sub-algebra K_m is *self-normalized*, that is, $\mathrm{Nor}K_m = K_m$, and the sub-algebra K_1 is *terminal* in the sense that the normalizer of any of its characteristic ideals is not identical to K_1. Arrows indicate the embedding of sub-algebras. Each self-normalized sub-algebra is like a root into which certain branches of ideals may enter forming a "cluster of ideals" with a common root. A normalized optimal system ΘL can thereby be depicted in the form of a "thicket of ideals", that is, the combination of all the clusters of ideals belonging to it.

Below, as an example, we consider the process of constructing the optimal system ΘL_{11} in the case of the Lie algebra L_{11} with a fixed basis (2.2).

The table of commutators for L_{11} is written out in the form of Table 2 where, for brevity, the basis elements X_i of (2.2) are replaced by their numbers i and the symbol $-i$ replaces $-X_i$.

The following composite series is immediately picked out and recorded using Table 2

$$0\subset\{X_1,X_2,X_3\}\subset\{X_1,\dots,X_6\}\subset\{X_1,\dots,X_9\}\subset\{X_1,\dots,X_{10}\}\subset L_{11} \tag{4.1}$$

The group of internal automorphisms A is generated by the basis (11×11)-matrices $A_i(a_i)$, which depend on the parameters a_i $(i=1,\dots,11)$. The result of the action of these matrices on the column vector $x=(x^i)$ from the coordinates of a common element $X=x^iX_i$ is shown in Table 3 which has been constructed according to the following rules. A subdivision of the vector x into subvectors (projections of x) is introduced in accordance with the factors of the composite series (4.1)

$$p_1=(x^1,x^2,x^3),\quad p_2=(x^4,x^5,x^6),\quad p_3=(x^7,x^8,x^9),\quad p_4=x^{10},\quad p_5=x^{11}$$

The parameters of the generating matrices $A_i(a_i)$ are grouped in a similar manner

$$e_1=(a_1,a_2,a_3),\quad e_2=(a_4,a_5,a_6),\quad e_3=(a_7,a_8,a_9),\quad e_4=a_{10},\quad e_5=a_{11}$$

The following notation is introduced for products of automorphisms

$$T=A_1A_2A_3,\quad \Gamma=A_4A_5A_6,\quad (S\otimes E_3)\times E_2=A_7A_8A_9,\quad A_{10},A_{11}$$

Table 2

	1	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	–3	2	0	1
2	0	0	0	0	0	0	3	0	–1	0	2
3	0	0	0	0	0	0	–2	1	0	0	3
4	0	0	0	0	0	0	0	–6	5	–1	0
5	0	0	0	0	0	0	6	0	–4	–2	0
6	0	0	0	0	0	0	–5	4	0	–3	0
7	0	–3	2	0	–6	5	0	–9	8	0	0
8	3	0	–1	6	0	–4	9	0	–7	0	0
9	–2	1	0	–5	4	0	–8	7	0	0	0
10	0	0	0	1	2	3	0	0	0	0	10
11	–1	–2	–3	0	0	0	0	0	0	–10	0

Table 3

	p_1'	p_2'	p_3'	p_4'	p_5'
T	$p_1+e_1p_5+e_1\wedge p_3$	p_2	p_3	p_4	p_5
Γ	$p_1-e_2p_4$	$p_2+e_2\wedge p_3$	p_3	p_4	p_5
S	Sp_1	Sp_2	Sp_3	p_4	p_5
A_{10}	$p_1+e_4p_2$	p_2	p_3	$p_4+e_4p_5$	p_5
A_{11}	e_5p_1	p_2	p_3	e_5p_4	p_5
ε_1	$-p_1$	$-p_2$	p_3	p_4	p_5
ε_2	p_1	$-p_2$	p_3	$-p_4$	p_5

Here, S is a general (3×3)-matrix of rotations in a three-dimensional space R^3, E_k are unit $(k\times k)$-matrices and the action of the remaining matrices has been explicitly indicated. Corresponding projections of the transformed vector $x'=Ax$ are denoted by a prime. The discrete automorphisms $\mathscr{E}_1$, $\mathscr{E}_2$, induced by the involutions I_1 and I_2 from Section 2 are added in Table 3. The symbol $\wedge$ denotes a conventional vector product in R^3.

The expansion $L_{11}=J\oplus N$, is used in the construction of ΘL_{11}, where

$$J=\{X_1,\dots,X_6\},\quad N=\{X_7,X_8,X_9,X_{10},X_{11}\}$$

It is obvious from Table 3 that a corresponding expansion of the augmented group $A=A_JA_N$ (without taking account of the involutions $\mathscr{E}_i$) holds with $A_J=TT$ and $A_N=SA_{10}A_{11}$.

The normalized optimal system $\Theta_{A_N}N$ has to be constructed during the first stage. Use of the composite series for N

$$0\subset\{X_7,X_8,X_9\}\subset\{X_7,X_8,X_9,X_{10}\}\subset N$$

enables one once again to apply the expansion procedure $N=J^1\oplus N^1$, where $J^1=\{X_7,\ X_8,\ X_9\}$ and $N^1=\{X_{10},\ X_{11}\}$. Here, the action of the augmented five-parameter group A_N^1 is described by the block in Table 3 which corresponds to the rows S, A_{10} and A_{11} and columns p_3', p_4', p_5'. The expansion of the group $A_N^1=A_{J^1}A_{N^1}$ holds with $A_{J^1}=S$ and $A_{N^1}=A_{10}$.

The constructions of $\Theta_{A_N}N$ is carried out in two substages. In the first substage, it is necessary to find the normalized optimal system $\Theta_{A_{N^1}}N^1$ of sub-algebras which is equivalent to the construction of the optimal system $\Theta_{G^1}(\zeta)$ of matrices

$$\zeta=\begin{Vmatrix}x^{10} & x^{11}\\ y^{10} & y^{11}\end{Vmatrix}$$

with respect to the action of the group $G^1=A_{10}B_2$. Since the rank $R(\zeta)$ of the matrix ζ is an invariant of the group G^1, the construction should be carried out using values of this rank. If $R(\zeta)=2$, the matrix ζ is reduced to a unit matrix by the obvious B_2-transformations. In the case $R(\zeta)=1$, when the matrix ζ consists of a single row, there is an "a priori" possibility of two reduced forms of this matrix: $(\beta,1)$ and $(1,0)$. However, the first of these is reduced to $(0,1)$ by means of the automorphism A_{10} (see Table 3). Finally, when $R(\zeta)=0$, the matrix ζ is a null matrix. So the first substage yields the optimal system $\Theta_{A_{N^1}}N^1=\{N_p\mid p=1,2,3,4\}$

$$N_1=\{X_{10},X_{11}\},\quad N_2=\{X_{11}\},\quad N_3=\{X_{10}\},\quad N_4=\{0\}\tag{4.2}$$

In the second substage we consider (5×5)-matrices with a block structure

$$\eta=\begin{Vmatrix}\eta^1 & \zeta\\ \eta^2 & 0\end{Vmatrix}$$

where ζ is one of the submatrices corresponding to (4.2) and the block η^2 follows after the first non-null row in ζ. Here, a new discrete invariant $R(\eta^2)$, arises which is equal to the rank of the (3×3)-submatrix η^2 for which the values 3, 2, 1, 0 are possible.

If $R(\eta^2)=3$, the submatrix η^2 is reduced to a unit matrix by means of the B-transformations while the submatrix η^1 is reduced to a null matrix. Together with (4.2), this yields the subalgebras

$$N_5=\{X_7,X_8,X_9,X_{10},X_{11}\},\quad N_6=\{X_7,X_8,X_9,X_{11}\},$$
$$N_7=\{X_7,X_8,X_9,X_{10},\},\quad N_8=\{X_7,X_8,X_9\}$$

An *additional condition* is subsequently used: if a single rotational operator occurs in the basis of a sub-algebra, it is reduced to X_7 by means of the automorphism S and the B-transform.

It is readily verified that the value $R(\eta^2)=2$ is found in contradiction with the equations of the sub-algebra (ES). Let $R(\eta^2)=1$. Then, the subvector $\eta^2=(x^7,\ x^8,\ x^9)$, according to the additional condition, is reduced to $\eta^2=(1,\ 0,\ 0)$. After the obvious B-transformations and use of the ES, it turns out that the submatrix η^1 is a null matrix in all cases of (4.2).

This yields the sub-algebras

$$N_9=\{X_7, X_{10}, X_{11}\},\ N_{10}=\{X_7, X_{11}\},\ N_{11}=\{X_7, X_{10}\},\ N_{12}=\{X_7\}$$

Finally, with $R(\eta^2)=0$ the matrix η^2 is a null matrix. In this case, it follows from the ES that the rank $R(\eta^1)$ of the submatrix η^1 cannot be equal to two. If $R(\eta^1)=0$, that is, $\eta^1=0$, then the sub-algebras (4.2) are obtained. If, however, $R(\eta^1)=1$, then the non-null row in η^1, according to the additional condition, reduces to $(\beta,\ 0,\ 0)$, where $\beta\neq 0$. Then, in the case of its combination with N_1 from the ES, it follows that a non-null row η^1 can be found in one row only with X_{11} which yields the sub-algebra $N_{13}=\{X_{10},\ X_7+\alpha X_{11}\}$. Similarly, with N_2, one obtains the sub-algebra $N_{14}=\{X_7+\alpha X_{11}\}$, and, with N_3, the sub-algebra $\{\beta X_7+X_{10}\}$ which is reduced to $N_{15}=\{X_7+X_{10}\}$ by the automorphism A_{11}. Finally, with N_4, one obtains the sub-algebra N_{13} which is already known.

At this point, the first stage in the construction of ΘL_{11} is concluded and the optimal system $\Theta_{A_N}N$ has been calculated. It contains 15 representatives, two of which form one-parameter series. The result of this construction is shown in Table 4 of sub-algebras N_p $(p=1,\ \ldots,\ 15)$, where the bases of the sub-algebras are only written symbolically using the numbers of the corresponding operators of the basis (2.2). Here, the symbol $7+\alpha 11$ denotes $X_7+\alpha X_{11}$ and so on.

The resulting optimal system $\Theta_{A_N}N$ is normalized. The numbers p of the normalizers of the sub-algebras N_p in N are indicated in the third column of Table 4 where self-normalized sub-algebras are labelled with the symbol $=p$. The stabilizers A_p of the matrices η_p, which correspond to the sub-algebras N_p are shown in the final column. They are directly found from Table 3. Here S_1 is the matrix for rotation about the first axis and the automorphism which acts on the (3×3)-matrix M according to the formula $\Sigma M=S^{-1}MS$. is denoted by the symbol Σ.

The thicket of ideals which corresponds to Table 4 is shown in Fig. 1. Here, it is very obvious that the separation of the sub-algebra {7} from the series {7+α11} is motivated by the difference in the normalizers in N. As far as the separation of the sub-algebras {7, 10} is concerned, this is

Table 4

p	Basis N_p	$\mathrm{Nor}_N N_p$	A_p
1	7, 8, 9, 10, 11	= 1	$TT\Sigma\varepsilon_1$
2	7, 8, 9, 11	= 2	$TTA_{11}\Sigma\varepsilon_1\varepsilon_2$
3	7, 10, 11	= 3	$TTS_1\varepsilon_1$
4	10,7 + α11 (α ≠ 0)	3	$TTS_1\varepsilon_1$
5	7, 11	= 5	$TTS_1A_{11}\varepsilon_1\varepsilon_2$
6	7 + α11 (α ≠ 0)	5	$TTS_1A_{11}\varepsilon_1\varepsilon_2$
7	7, 8, 9, 10	1	$TTA_{10}\Sigma\varepsilon_1$
8	7, 8, 9	1	$TTA_{10}A_{11}\Sigma\varepsilon_1\varepsilon_2$
9	7, 10	3	$TTS_1A_{10}\varepsilon_1$
10	7	3	$TTS_1A_{10}A_{11}\varepsilon_1\varepsilon_2$
11	7 + 10	9	$TTS_1A_{10}\varepsilon_1$
12	10, 11	1	$TTS\varepsilon_1$
13	11	2	$TTSA_{11}\varepsilon_1\varepsilon_2$
14	10	1	$TTSA_{10}\varepsilon_1$
15	0	1	$A\varepsilon_1\varepsilon_2$

induced by the difference between the normalizers of these sub-algebras in the whole of the Lie L^{11} algebra. Self-normalized sub-algebras are picked out by putting them in double frames. The numbers r denote the dimensions of sub-algebras.

In the second stage of the construction of ΘL_{11}, the matrices

$$\xi_p = \begin{Vmatrix} \xi^1 & \xi^2 & \eta_p \\ \xi^3 & \xi^4 & 0 \\ \xi^5 & 0 & 0 \end{Vmatrix}$$

are successively considered for each $p = 1, \ldots, 15$. These consist of 11 columns and $r \le 11$ rows which are subdivided into blocks in accordance with the composite series (4.1). The block submatrices ξ^j each have 3 columns and ≤ 3 rows and η_p is a matrix which corresponds to the sub-algebra N_p from Table 4.

The second stage involves the construction of the subsystem $\Theta_{G_p}\{\xi_p\}$ of matrices ξ_p for each $p = 1, \ldots, 15$ that satisfy the conditions of the sub-algebra (CS). This subsystem is optimal with respect to the action of the group $G_p = A_p B_p$, where A_p are the stablilizers indicated in Table 3 and $B_p \subset B$ is a subgroup of the group of B-transforms which leave the submatrix η_p at the site.

Here, it is inadvisable to describe every detail of the execution of the second stage in view of its monotony and tediousness for the reader as well as the fact that it is practically impossible here, since such a description requires many pages. It is pertinent solely to note the following main features of its construction.

The ranks of the submatrices $R(\xi^4)$ and $R(\xi^5)$ are invariants of any G_p-transformations. Since each of them can take four values 3, 2, 1 and 0, sixteen dissimilar forms of constructing the matrices ξ_p arise. Each of the matrices ξ^4, ξ^5 is first reduced to certain standard forms (there is a certain arbitrariness in the choice of these) by means of G_p-transformations. Those of these forms which do not satisfy the ES with η_p are then discarded. The remaining forms serve as a basis for the final reduction of the matrix ξ_p using G_p-transformations and separating out the actual representatives in the required optimal system. The property of normalization is traced during this process by means of a sequence of analyses from the smaller dimensions to the larger and systematic calculation of the normalizers of the resulting sub-algebras.

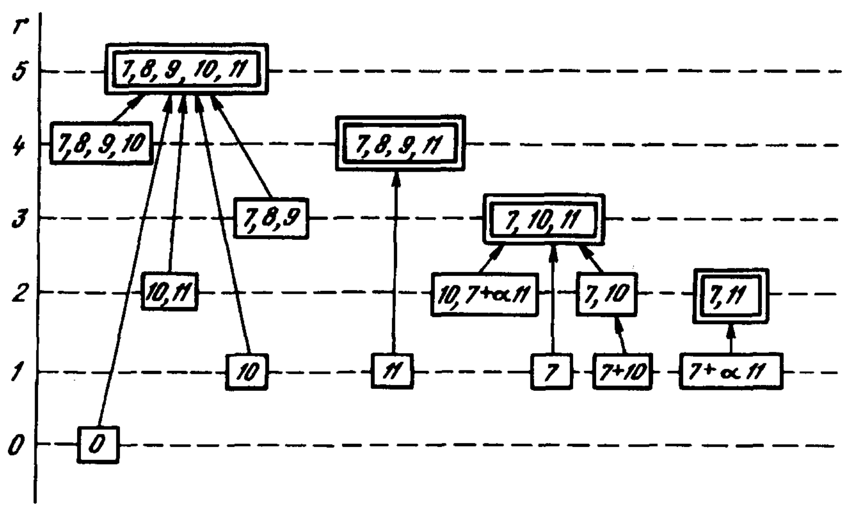

Fig. 1.

As a result of carrying out the second stage, the normalized optimal system ΘL_{11} of the sub-algebras of the L_{11} Lie group with the basis (2.2) is obtained. This is presented in the appendix.

5. INVARIANT SUBMODELS OF RANK THREE

In the case of the PODMODELI program, the ordering of the submodels by the introduction of certain way of labelling them is important. In the case of gas dynamics, a notation for the submodels with a two lower-case letter matrix index is proposed. The "large models" from Table 1 are denoted by

$$\begin{pmatrix}4\\k\end{pmatrix},\ k = 1,\dots,13.$$

The submodels of rank three of a "large model"

$$\begin{pmatrix}4\\k\end{pmatrix}$$

are denoted by the index

$$\begin{pmatrix}4 & 3\\k & l\end{pmatrix},\ l = 1,2,\dots,$$

while the index

$$\begin{pmatrix}4 & 2\\k & m\end{pmatrix},\ m = 1,2,\dots$$

is used for submodels of rank 2 and so on, where the numbers $l, m, \dots$ are taken from the corresponding tables.

In this section we present a preliminary description of the submodels

$$\begin{pmatrix}4 & 3\\1 & l\end{pmatrix},\ l = 1,\dots,13,$$

which are obtained with respect to single-parameter subgroups which correspond to the sub-algebras $L_{1,l}$ from Table 6 (see the appendix). The submodels with respect to the sub-algebras with numbers $l = 7, \dots, 13$. which do not contain a rotational operator, are written in Cartesian coordinates. The rotational operator X_7 is contained in the sub-algebras with $l = 1, \dots, 6$ and it is more convenient to represent the corresponding submodels in cylindrical coordinates which are introduced by the following relationships.

The independent variables will be t, x, r and θ where

$$y = r\cos\theta,\ z = r\sin\theta;\ r = \sqrt{y^2+z^2},\ \theta = \operatorname{arctg}\frac{z}{y} \tag{5.1}$$

The components of the velocity vector $\mathbf{u} = \mathbf{u}_c = (u_c,\ \upsilon_c,\ w_c)$ are introduced by the formulae

$$\begin{aligned} &u_c = u, \quad \upsilon_c = \upsilon\cos\theta + w\sin\theta, \quad w_c = -\upsilon\sin\theta + w\cos\theta \\ &u = u_c, \quad \upsilon = \upsilon_c\cos\theta - w_c\sin\theta, \quad w = \upsilon_c\sin\theta + w_c\cos\theta \end{aligned} \tag{5.2}$$

Here, υ_c is the radial component of the velocity (in the y, z plane) and w_c is the peripheral component of the velocity vector. The notation used for density and pressure remains as before: ρ and p.

The initial system of equations of gas dynamics (2.1) takes the following form in cylindrical coordinates:

$$D_c u_c + \rho^{-1}\nabla_c p = r^{-1}(0, w_c^2, -\upsilon_c w_c)$$

$$D_c\rho+\rho\,\mathrm{div}_c\,\mathbf{u}=0,\quad D_c p+A\,\mathrm{div}_c\,\mathbf{u}=0 \tag{5.3}$$

where

$$\nabla_c=(\partial_x,\partial_r,r^{-1}\partial_\theta),\quad \mathrm{div}_c\,\mathbf{u}=\mathbf{u}_{cx}+\upsilon_{cr}+r^{-1}\upsilon_c+r^{-1}w_{c\theta}$$

$$D_c=\partial_t+\mathbf{u}_c\nabla_c=\partial_t+\mathbf{u}_c\partial_x+\upsilon_c\partial_r+r^{-1}w_c\partial_\theta$$

In the cylindrical coordinates (5.1), (5.2), the basis operators X_7 and X_{11} involved in the construction of submodels of rank three will be $X_7=\partial_\theta$, $X_{11}=t\partial_t+x\partial_x=r\partial_r$, while the remaining coordinates do not change.

The general rule for constructing invariant H-solutions is as follows [2]. The independent invariants of group H are found which, generally speaking, are chosen with a certain arbitrariness. Relationships are then established between these invariants of such a kind that the invariants—the required quantities are designated as functions of the invariants—ndependent variables. These functions will also be new required functions in the submodel—the factor system E/H. It is formulated after the expressions for the initial required quantities, obtained from the relationships between the invariants, have been substituted into the initial system E.

A list of the sets of invariants subsequently used for all of the 13 submodels

$$\begin{pmatrix}4 & 3\\ 1 & k\end{pmatrix},$$

is given in Table 5 where the number k is shown in the first column. In the second column, it is recalled that submodels with $k=1,\ldots,6$ are treated in cylindrical coordinates (C) while submodels with $k=7,\ldots,13$ are treated in Cartesian coordinates (D). The operators of the single-parameter subgroups H_k, from which the invariant H_k-solutions are constructed, are presented in the third column using the notation of (2.2). The invariant independent variables are written out in the fourth column, and those of them which differ from the initial variables are given the subscript 1. The notation $\mathbf{U}=(U,\ V,\ W)$, ρ, p is adopted for the new required invariant magnitudes of the velocity vector, density and pressure. Here, in all submodels $V=\upsilon_c$, $W=w_c$ (in C coordinates) or $V=\upsilon$, $W=w$ (in D coordinates), and ρ and p are invariants. For this reason, only the expression for the invariant component of U is explicitly given in the fifth column.

The actual factor systems for all submodels of rank three according to Table 5 are presented below. As was stated at the beginning of this paper, the PODMODELI program also presumes a general analysis ("dressing") of the resulting submodels. However, this task is outside the scope of this paper and must be the subject of subsequent publications. Here, we shall merely point out the descriptive characteristics of the corresponding motions of the gas and note their purely geometrical structure, which is determined by the level lines of the invariants. In the case of invariant submodels of rank three, these lines play the same role as points in the event space $R^4(t,\ \mathbf{x})$ in the case of the "large model".

A curve $\mathcal{L}$ in the space $R^4(t,\ \mathbf{x})$ along which the independent variables in the factor system (invariants) maintain a constant value is referred to as a level line of the invariants. The set of all level lines of the invariants is denoted by the symbol $\{\mathcal{L}\}$.

The enumeration proceeds in the reverse order with respect to Table 5 on proceeding from the simpler to the more complex submodels. An invariant velocity vector is everywhere denoted by $\mathbf{U}=(U,V,W)$.

The *submodel*

$$\begin{pmatrix}4 & 3\\ 1 & 13\end{pmatrix}$$

describes the two-dimensional motion of a gas. A representation of the solution is

$$(\mathbf{u},\rho,p)=(\mathbf{U},\rho,p)(t,y,z) \tag{5.4}$$

Table 5

k	Coordinate system	Operator	Invariant independent variables	Invariant component U
1		$\beta X_4 + X_7 + \alpha X_{11}$ $(\alpha \neq 0)$	$x_1 = \frac{x}{t} - \beta\theta, r_1 = \frac{r}{t}, \theta_1 = \theta - \frac{1}{\alpha}\ln t$	$u = u_c - \beta\theta$
2		$\beta X_4 + X_7$	$t, x_1 = x - \beta t\theta, r$	
3	C	X_7	t, x, r	$U = u_c$
4		$X_1 + X_7$	$t, x_1 = x - \theta, r$	
5		$\beta X_4 + X_7 + \beta X_{10}$	$t_1 = t - \beta\theta, x_1 = x - \frac{1}{2}t^2, r$	$U = u_c - t$
6		$X_7 + X_0$	$t_1 = t - \theta, x, r$	$U = u_c$
7		$\beta X_4 + X_{11}$	$x_1 = \frac{x}{t} - \beta\ln t, y_1 = \frac{y}{t}, z_1 = \frac{z}{t}$	$U = u - \frac{x}{t}$
8		X_{11}	$x_1 = \frac{x}{t}, y_1 = \frac{y}{t}, z_1 = \frac{z}{t}$	$U = u$
9	D	$X_4 + X_{10}$	$x_1 = x - \frac{1}{2}t^2, y, z$	$U = u - t$
10		X_{10}	x, y, z	$U = u$
11		$X_3 + X_4$	$t, x_1 = x - tz, y$	$U = u - z$
12		X_4	t, y, z	$U = u - \frac{x}{t}$
13		X_1	t, y, z	$U = u$

The factor system of the equations of the submodel is

$$D_1 U = 0, \quad D_1 V + \rho^{-1} p_y = 0, \quad D_1 W + \rho^{-1} p_z = 0 \tag{5.5}$$
$$D_1\rho + \rho(V_y + W_z) = 0, \quad D_1 p + A(V_y + W_z) = 0$$

where $D_1 = \partial_t + V\partial_y + W\partial_z$.

In particular, with $U = 0$, this submodel describes the *plane-parallel* motions of a gas; $\{\mathcal{L}\}$ is a family of lines parallel to the x-axis.

The *submodel*

$$\begin{pmatrix} 4 & 3 \\ 1 & 12 \end{pmatrix}$$

describes the Galilean-invariant motion of a gas. A representation of the solution is

$$u = \frac{x}{t} + U(t, y, z), \quad (\upsilon, w, \rho, p) = (V, W, \rho, p)(t, y, z) \tag{5.6}$$

The factor system is

$$D_1(tU) = 0, \quad D_1 V + \rho^{-1} p_y = 0, D_1 W + \rho^{-1} p_z = 0 \tag{5.7}$$
$$D_1(t\rho) + t\rho(V_y + W_z) = 0, D_1(tp) + tA(V_y + W_z) = 0$$

where $D_1 = \partial_t + V\partial_y + W\partial_z$.

Here, $\mathcal{L}$ also consists of straight lines which are parallel to the x-axis.

It is of interest to compare the factor systems (5.5) and (5.7), in particular, in the case of a polytropic gas $(A = \gamma p)$.

The *submodel*

$$\begin{pmatrix} 4 & 3 \\ 1 & 11 \end{pmatrix}$$

describes the shear motions of a gas. A representation of the solution is

$$u = z + U(t, x_1, y), (\upsilon, w, \rho, p) = (V, W, \rho, p)(t, x_1, y), x_1 = x - tz \tag{5.8}$$

The factor system is

$$\begin{aligned} &D_1 U + \rho^{-1} p_{x_1} = -W, \quad D_1 V + \rho^{-1} p_y = 0 \\ &D_1 W - t\rho^{-1} p_{x_1} = 0; D_1 \rho + \rho(U_{x_1} - tW_{x_1} + V_y) = 0 \\ &D_1 p + A(U_{x_1} - tW_{x_1} + V_y) = 0 \end{aligned} \tag{5.9}$$

where $D_1 = \partial_t + (U = tW)\partial_{x_1} + V\partial_y$.

The lines of $\mathscr{L}$ form a family of straight lines which are parallel to the (x, y)-plane, the angular coefficient of which depends on t.

The *submodel*

$$\begin{pmatrix} 4 & 3 \\ 1 & 10 \end{pmatrix}$$

describes the (steady) flows of a gas, a representation of the solution is

$$(\mathbf{u}, \rho, p) = (\mathbf{U}, \rho, p)(x, y, z) \tag{5.10}$$

The factor system is

$$D_1 \mathbf{U} + \rho^{-1} \nabla p = 0, \quad D_1 \rho + \rho \operatorname{div} \mathbf{U} = 0, \quad D_1 p + A \operatorname{div} \mathbf{U} = 0 \tag{5.11}$$

where $D_1 = U\partial_x + V\partial_y + W\partial_z$.

The lines of $\mathscr{L}$ form a family of straight lines which are parallel to the t-axis.

The *submodel*

$$\begin{pmatrix} 4 & 3 \\ 1 & 9 \end{pmatrix}$$

describes the steady flows of a gas in a homogeneous force field directed parallel to the x-axis. A representation of the solution is

$$u = t + U(x, y, z), (\upsilon, w, \rho, p) = (V, W, \rho, p)(x_1, y, z), \quad x_1 = x - t^2/2 \tag{5.12}$$

The factor system is

$$\begin{aligned} &D_1 U + \rho^{-1} p_{x_1} = -1, D_1 V + \rho^{-1} p_y = 0, D_1 W + \rho^{-1} p_z = 0 \\ &D_1 \rho + \rho(U_{x_1} + V_y + W_z) = 0, \quad D_1 p + A(U_{x_1} + V_y + W_z) = 0 \end{aligned} \tag{5.13}$$

where $D_1 = U\partial_x + V\partial_y + W\partial_z$.

The lines of $\mathscr{L}$ form a family of congruent parabolae lying in planes which are parallel to the (y, z)-plane.

The *submodel*

$$\begin{pmatrix} 4 & 3 \\ 1 & 8 \end{pmatrix}$$

describes the conically self-similar motions of a gas. A representation of the solution is

$$(\mathbf{u},\rho,p)=(\mathbf{U},\rho,p)(\mathbf{x}_1),\ \mathbf{x}_1=\mathbf{x}/t \tag{5.14}$$

With the operator $D_1=(U-x_1)\partial_{x_1}+(V-y_1)\partial_{y_1}+(W-z_1)\partial_{z_1}$, the factor system has the form of (5.11) where ∇ has to be replaced by $\nabla_1=(\partial_{x_1},\ \partial_{y_1},\ \partial_{z_1})$ and $\mathrm{div}U$ has to be replaced by $\mathrm{div}_1U=U_{x_1}+V_{y_1}+W_{z_1}$. The family $\{\mathcal{L}\}$ is a set of half-lines (rays) which emerge from the origin of coordinates.

The *submodel*

$$\begin{pmatrix}4 & 3\\ 1 & 7\end{pmatrix}$$

describes the quasiconical (or generalized conical) motions of a gas. A representation of the solution is

$$u=\frac{x}{t}+U(x_1,y_1,z_1),\quad (v,w,\rho,p)=(V,W,\rho,p)(x_1,y_1,z_1) \tag{5.15}$$

$$x_1=\frac{x}{t}-\beta\ln t,\quad y_1=\frac{y}{t},\ z_1=\frac{z}{t}$$

The factor system is written in the form

$$\begin{aligned}&D_1U+\rho^{-1}p_{x_1}=-U,\ D_1V+\rho^{-1}p_{y_1}=0\\ &D_1W+\rho^{-1}p_{z_1}=0;\ D_1\rho+\rho(U_{x_1}+V_{y_1}+W_{z_1}+1)=0\\ &D_1p+A(U_{x_1}+V_{y_1}+W_{z_1}+1)=0\end{aligned} \tag{5.16}$$

where $D_1=(U-\beta)\partial_{x_1}+(V-y_1)\partial_{y_1}+(W-z_1)\partial_{z_1}$.

The family $\{\mathcal{L}\}$ forms quasirays which are plane curves which emerge from the origin of coordinates. Their projections onto the space $R^3(\mathbf{x})$, which are obtained by eliminating the time t from formulae (5.15) provides a visual representation of these curves. A typical projection, which has an equation of the form $x=ar+br\ln r$ in its (x, r)-plane is shown in Fig. 2.

The remaining submodels are treated in cylindrical coordinates using (5.2) and (5.3).

The submodel

$$\begin{pmatrix}4 & 3\\ 1 & 6\end{pmatrix}$$

describes the rotational motions of a gas. A representation of the solution is

$$(\mathbf{u}_c,\rho,p)=(\mathbf{U},\rho,p)(t_1,x,r),\quad t_1=t-\theta \tag{5.17}$$

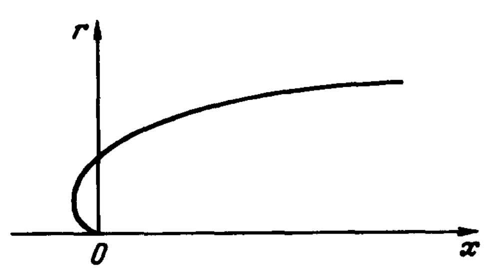

Fig. 2.

The factor system is

$$
\begin{aligned}
&D_1U+\rho^{-1}p_x=0,\ D_1V+\rho^{-1}p_r=+r^{-1}W^2\\
&D_1W-r^{-1}\rho^{-1}p_{t_1}=-r^{-1}VW\\
&D_1\rho+\rho(U_x+V_r+r^{-1}V-r^{-1}W_{t_1})=0\\
&D_1p+A(U_x+V_r+r^{-1}V-r^{-1}W_{t_1})=0
\end{aligned}
\tag{5.18}
$$

where $D_1=(l-r^{-1}W)\partial_{t1}+U\partial_x+V\partial_r$.

Circles with their centres on the x-axis, lying in planes which are perpendicular to this axis, are the projections of the lines of $\mathcal{L}$ onto the space $R^3(\mathbf{x})$.The lines of $\mathcal{L}$ themselves are spirals with a constant step "wound" on circular cylinders with an x-axis.

The *submodel*

$$\begin{pmatrix}4&3\\1&5\end{pmatrix}$$

describes the generalized rotational motions of a gas in a homogeneous force field directed parallel to the x-axis. A representation of the solution is

$$
\begin{aligned}
&u_c=t+U(t_1,x_1,r),\ (\upsilon_c,w_c,\rho,p)=(V,W,\rho,p)(t_1,x_1,r)\\
&t_1=t-\beta\theta,\quad x_1=x-t^2/2
\end{aligned}
\tag{5.19}
$$

The factor system is written in the form

$$
\begin{aligned}
&D_1U+\rho^{-1}p_{x_1}=-1,\ D_1V+\rho^{-1}p_r=+r^{-1}W^2\\
&D_1W-\beta r^{-1}\rho^{-1}p_{t_1}=-r^{-1}VW\\
&D_1\rho+\rho(U_{x_1}+V_r+r^{-1}V-\beta r^{-1}W_{t_1})=0\\
&D_1p+A(U_{x_1}+V_r+r^{-1}V-\beta r^{-1}W_{t_1})=0
\end{aligned}
\tag{5.20}
$$

where $D_1=(1-\beta r^{-1}W)\partial_{t_1}+U\partial_{x_1}+V\partial_r$.

The complex form of the lines of $\mathcal{L}$ in the case of this submodel admits of a simple interpretation if these lines are considered in a system of coordinates which moves according to the law $x=t^2/2$ in the direction of the x-axis. The family $\{\mathcal{L}\}$ then appears to be the same as in the submodel

$$\begin{pmatrix}4&3\\1&6\end{pmatrix},$$

with a spiral step equal to $2\pi\beta$.

The submodel

$$\begin{pmatrix}4&3\\1&4\end{pmatrix}$$

describes the spiral motions of a gas. A representation of the solution is

$$(\mathbf{u}_c,\rho,p)=(\mathbf{U},\rho,p)(t,x_1,r),\ x_1=x-\theta \tag{5.21}$$

The factor system has the form

$$
\begin{aligned}
&D_1U+\rho^{-1}p_{x_1}=0,\ D_1V+\rho^{-1}p_r=+r^{-1}W^2\\
&D_1W-r^{-1}\rho^{-1}p_{x_1}=-r^{-1}VW\\
&D_1\rho+\rho(U_{x_1}-r^{-1}W_{x_1}+V_r+r^{-1}V)=0\\
&D_1p+A(U_{x_1}-r^{-1}W_{x_1}+V_r+r^{-1}V)=0
\end{aligned}
\tag{5.22}
$$

where $D_1=\partial_t+(V-r^{-1}W)\partial_{x_1}+V\partial_r$.

Here, $\{\mathscr{L}\}$ is a family of straight lines parallel to the t-axis which are projected onto the space R^3 in the form of a set of spirals with a constant step size, "wound" onto circular cylinders with an x-axis.

The *submodel*

$$\begin{pmatrix}4 & 3\\ 1 & 3\end{pmatrix}$$

describes the rotationally symmetric motions of a gas. A representation of the solution is

$$(\mathbf{u}_c,\rho,p)=(\mathbf{U},\rho,p)(t,x,r) \tag{5.23}$$

The factor system is

$$
\begin{aligned}
&D_1U+\rho^{-1}p_x=0,\ D_1V+\rho^{-1}p_r=+r^{-1}W^2,\ D_1W=-r^{-1}VW\\
&D_1\rho+\rho(U_x+V_r+r^{-1}V)=0,\ D_1p+A(U_x+V_r+r^{-1}V)=0
\end{aligned}
\tag{5.24}
$$

where $D_1=\partial_t+U\partial_x+V\partial_r$.

The lines $\mathscr{L}$ form a family of circles lying in planes perpendicular to the x-axis (in R^3) with their centres on this axis.

The particular solutions of (5.4) with $W=0$ describe the axially symmetric motions of a gas.

The *submodel*

$$\begin{pmatrix}4 & 3\\ 1 & 2\end{pmatrix}$$

describes the generalized rotationally symmetric motions of a gas.

A representation of the solution is

$$
\begin{aligned}
&u_c=\beta\theta+U(t,x_1,r),\ (\upsilon_c,w_c,\rho,p)=(V,W,\rho,p)(t,x_1,r)\\
&x_1=x-\beta t\theta
\end{aligned}
\tag{5.25}
$$

The factor system is

$$
\begin{aligned}
&D_1U+\rho^{-1}p_{x_1}=-\beta r^{-1}W,\ D_1V+\rho^{-1}p_r=+r^{-1}W^2\\
&D_1W-\beta tr^{-1}\rho^{-1}p_{x_1}=-r^{-1}VW\\
&D_1\rho+\rho(U_{x_1}-\beta tW_{x_1}+V_r+r^{-1}V)=0\\
&D_1p+A(U_{x_1}-\beta tW_{x_1}+V_r+r^{-1}V)=0
\end{aligned}
\tag{5.26}
$$

where $D_1=\partial_t+(U-\beta tr^{-1}W)\partial_{x_1}+V\partial_r$.

The projections of the curves of $\mathscr{L}$ onto the space $R^3(\mathbf{x})$ have, as in the case of the submodel

$$\begin{pmatrix}4 & 3\\ 1 & 4\end{pmatrix}$$

the form of spirals "wound" on circular cylinders with an x-axis. Here, however, the step size of the spirals depends on the time t.

The *submodel*

$$\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$$

describes the quasiconical spiral motions of a gas. A representation of the solution is

$$\begin{aligned} &u_c = \beta\theta + U(x_1, r_1, \theta_1), (\upsilon_c, w_c, \rho, p) = (V, W, \rho, p)(x_1, r_1, \theta_1) \\ &x_1 = x/t - \beta\theta, r_1 = r/t, \theta_1 = \theta - \alpha^{-1}\ln t \end{aligned} \tag{5.27}$$

The factor system is

$$\begin{aligned} &D_1U + \rho^{-1}p_{x_1} = -\beta r_1^{-1}W, D_1V + \rho^{-1}p_{r_1} = +r_1^{-1}W^2 \\ &D_1W - \beta t r_1^{-1}\rho^{-1}p_{x_1} + r_1^{-1}\rho^{-1}p_{\theta_1} = -r_1^{-1}VW \\ &D_1\rho + \rho(U_{x_1} - \beta r_1^{-1}W_{x_1} + V_{r_1} + r_1^{-1}V + r_1^{-1}W_{\theta_1}) = 0 \\ &D_1p + A(U_{x_1} - \beta r_1^{-1}W_{x_1} + V_{r_1} + r_1^{-1}V + r_1^{-1}W_{\theta_1}) = 0 \end{aligned} \tag{5.28}$$

where $D_1 = (U - x_1 - \beta r_1^{-1}W)\partial_{x_1} + (V - r_1)\partial_{r_1} + (r_1^{-1}W - \alpha^{-1})\partial_{\theta_1}$.

The projection of the lines of $\mathscr{L}$ onto the space $R^3(\mathbf{x})$ are spirals "wound" onto surfaces of revolution of curves of the form shown in Fig. 2.

In the special case when $\beta = 0$, this submodel describes the conical spiral motions of a gas. In the case of these motions, the projections of $\mathscr{L}$ onto $R^3(\mathbf{x})$ are spirals "wound" on circular cones with a vertex at the origin of coordinates and with an x-axis.

I wish to thank the participants at the seminar on "Group analysis" (Institute of Hydrodynamics of the Siberian Branch of the Russian Academy of Science) S. V. Meleshko, S. V. Khabirov, A. A. Talyshev, A. P. Chupakhin, E. V. Mamontov and A. A. Cherevko with whom the contents of this paper were discussed.

The work was carried out with the financial support from the Russian Fund for Fundamental Research (93-013-17326).

APPENDIX. THE NORMALIZED OPTIMAL SYSTEM ΘL_{11}

The normalized optimal system of sub-algebras of the Lie algebra L_{11} with the basis (2.2), calculated using the algorithm in Section 4, is presented in Table 6. The sub-algebra representatives are denoted by a pair of numbers (r, i), where r is the dimensions and i is the serial number of a sub-algebra of dimensions r. The numbers r are given in front of each block containing sub-algebras of dimensions r. The serial numbers i are presented in the first column. The bases of the sub-algebras (r, i) are written out in abbreviated symbolic form in the second column (only the numbers of the corresponding basis vectors of (2.2) are written out as in Table 4 (see Section 4)). Here, the possible constraints on such parameters are indicated and the absence of such an indication means that the parameters can have any real values. The normalizers of the sub-algebras in L_{11} are presented in the third column. By virtue of the normalized character of the whole table, these are contained in the same table and are therefore indicated as necessary for a pair (r', i'). Here, the equals sign denotes that the corresponding sub-algebra is self-normalized. A superscript 0 indicates that the normalizer is contained in a series of sub-algebras and is obtained with a zero value of the parameter occurring in it.

Table 6

i	Basis	Nor
	r = 11	
1	1, 2, 3; 4, 5, 6; 7, 8, 9; 10; 11	= 11.1
	r = 10	
1	1, 2, 3; 4, 5, 6; 7, 8, 9; 11	= 10.1
2	1, 2, 3; 4, 5, 6; 7, 8, 9; 10	11.1
	r = 9	
1	1, 2, 3; 4, 5, 6; 7; 10; 11	= 9.1
2	1, 2, 3; 4, 5, 6; 7, 8, 9	11.1
	r = 8	
1	1, 2, 3; 7, 8, 9; 10; 11	= 8.1
2	1, 2, 3; 5, 6; α4 + 7; 10; β4 + 11	9.1
3	1, 2, 3; 4, 5, 6; 7 + α11; 10	9.1
4	1, 2, 3; 4, 5, 6; 7; 11	= 8.4
5	1, 2, 3; 4, 5, 6; 10; 11	11.1
	r = 7	
1	1, 2, 3; 7, 8, 9; 11	= 7.1
2	4, 5, 6; 7, 8, 9; 11	= 7.2
3	2, 3; 5, 6; 7; 10; 11	= 7.3
4	1, 2, 3; 4; 7; 10; 11	= 7.4
5	1, 2, 3; 5, 6; 10; β4 + 7 + α11	9.1
6	2, 3; 4, 5, 6; 7; 11	= 7.6
7	1, 2, 3; 5, 6; α4 + 7; β4 + 11	8.4
8	1, 2, 3; 4, 5, 6; 7 + α11	8.4
9	1, 2, 3; 7, 8, 9; 10	8.1
10	1, 2, 3; 5, 6; α4 + 7; 4 + 10	8.3°
11	1, 2, 3; 4, 5, 6; 7 + 10	8.3°
12	1, 2, 3; 5, 6; 10; α4 + 11	9.1
13	1, 2, 3; 4, 5, 6; 11	10.1
14	1, 2, 3; 4, 5, 6; 10	11.1
	r = 6	
1	1, 2, 3; 7; 10; 11;	7.4
2	2, 3; 5, 6; 10; 7 + α11 α ≠ 0	7.3
3	1, 2, 3; 4; 10; 7 + α11	7.4
4	1; 4, 5, 6; 7; 11	= 6.4
5	1, 2, 3; 4; 7; 11	= 6.5
6	2, 3; 5, 6; α4 + 7; β4 + 11	7.6
7	2, 3; 4, 5, 6; 7 + α11 α ≠ 0	7.6
8	1, 2, 3; 5, 6; β4 + 7 + α11	8.4
9	4, 5, 6; 7, 8, 9	7.2
10	1, 2, 3; 7, 8, 9	8.1
11	2, 3; 5, 6; 1 + 7; 10	7.5°°
12	2, 3; 5, 6; α1 + 7; 4 + 10	7.10°
13	2, 3; 5, 6; 7; 10	8.2°°
14	2, 3; 4, 5, 6; 1 + 7	7.8°
15	2, 3; 4, 5, 6; 7	8.4
16	1, 2, 3; 5, 6; 7 + 10	7.5°°
17	2, 3; 5, 6; 10, 11	7.3
18	1, 2, 3; 4; 10; α6 + 11 α ≠ 0	8.5
19	1, 2, 3; 4; 10; 11	9.1
20	1, 2, 3; 5, 6; α4 + 11	8.4
21	2, 3; 4, 5, 6; 11	7.6
22	1, 2, 3; 5, 6; 10	9.1
23	1, 2, 3; 5, 6; 4 + 10	8.3°
24	1, 2, 3; 4, 5, 6	11.1
	r = 5	
1	7, 8, 9; 10; 11	= 5.1
2	1; 4; 7; 10; 11	= 5.2
3	2, 3; 7; 10; 11	= 5.3
4	1, 2, 3; 10; β4 + 7 + α11	7.4
5	4, 5, 6; 7; 11	= 5.5
6	2, 3, 4; 7; 11	= 5.6
7	1; 5, 6; α4 + 7; β4 + 11	6.4
8	1, 2, 3; α4 + 7; β4 + 11	6.5
9	1; 4, 5, 6; 7 + α11	6.4
10	2, 3; 5, 6; β4 + 7 + α11 α ≠ 0	7.6
11	1, 2, 3; 4; 7 + α11 α ≠ 0	6.5
12	1, 2, 3; α4 + 7; 4 + 10	6.3°
13	2, 3; 5, 6; β4 + 7 β ≠ 0	8.4
14	2, 3; 5, 6; 7	9.1
15	1, 2, 3; 4; 7	9.1
16	1; 4, 3 + 5, 2 − 6; 7	= 5.16
17	2, 3; 5, 6; 1 + 7	8.3°
18	2, 3; 5, 6; β4 + 7 + β10 β ≠ 0	7.10°
19	2, 3; 5, 6; 7 + 10	7.5°°
20	1, 2, 3; 4; 7 + 10	6.3°
21	2, 3; 5; 10; β6 + 11	6.17
22	1, 2, 3; 10; 4 + β11	9.1
23	1, 2, 3; 10; 11	11.1
24	1; 4, 5, 6; 11	6.4
25	2, 3; α4 + 5, 6; β4 + 11 α ≠ 0	6.21
26	2, 3; 5, 6, β4 + 11	7.6
27	2, 3; 4, 6; β5 + 11	6.21
28	1, 2, 3; 6; β4 + 11 β ≠ 0	7.13
29	1, 2, 3; 4; 11	8.4
30	2, 3; α1 + 5, 6; 4 + 10 α ≠ 0	6.23
31	2, 3; 5, 6; 4 + 10	7.10°
32	2, 3; 1 + 5, 6; 10	6.22

Table 6 (*Continued*)

i	Basis	Nor
	$r = 5$	
33	$2, 3; 5, 6; 10$	8.2°°
34	$1, 2, 3; 6; 4 + 10$	7,14
35	$2, 3; 4, 5, 6$	8,4
36	$2, 3; 4, 5; 1 + 6$	6,24
37	$1, 2, 3; 5, 6$	9,1
	$r = 4$	
1	$7, 8, 9; 11$	= 4,1
2	$1; \alpha 4 + 7; 10; 11$	5.2
3	$2, 3; 10; 7 + \alpha 11$ $\alpha \neq 0$	5.3
4	$1; 4; 10; 7 + \alpha 11$	5.2
5	$5, 6; \alpha 4 + 7; \beta 4 + 11$	5.5
6	$1; 4; 7; 11$	= 4.6
7	$2, 3; \alpha 4 + 7; \beta 4 + 11$	5.6
8	$4, 5, 6; 7 + \alpha 11$ $\alpha \neq 0$	5.5
9	$1; 5, 6; \beta 4 + 7 + \alpha 11$	6.4
10	$2, 3; 4; 7 + \alpha 11$	5.6
11	$1, 2, 3; \beta 4 + 7 + \alpha 11$ $\alpha \neq 0$	6.5
12	$1, 2, 3; \beta 4 + 7$	7.4
13	$7, 8, 9; 10$	5.1
14	$2, 3; 7; 10$	6.1
15	$2, 3; 1 + 7; 10$	5.4°°
16	$2, 3; \alpha 1 + 7; 4 + 10$	5.12°
17	$4, 5, 6; 7$	6.4
18	$4, 5, 6; 1 + 7$	5.9°
19	$4, 3 + 5, 2 - 6; \alpha 1 + 7$	5.16
20	$1; 3 + 5, 2 - 6; \alpha 4 + 7$	5.16
21	$2, 3; 4; 1 + 7$	5.15
22	$1, 2, 3; \beta 4 + 7 + 10$	6,3°
23	$1; 4; 10; 11$	5.2
24	$2, 3; 10; \alpha 6 + 11$ $\alpha \neq 0$	6.17
25	$2, 3; 10; 11$	7.3
26	$4, 5, 6; 11$	7.2
27	$1; \alpha 4 + 5, 6; \beta 4 + 11$ $\alpha \neq 0$	5.24
28	$1; 5, 6; \beta 4 + 11$	6.4
29	$1; 4, 6; \alpha 5 + 11$	5.24
30	$2, 3; \alpha 4 + 6; \beta 4 + \sigma 5 + 11$	6.21
31	$2, 3; 4; \alpha 5 + \beta 6 + 11$ $\alpha^2 + \beta^2 \neq 0$	6.21
32	$2, 3; 4; 11$	7.6
33	$1, 2, 3; \beta 4 + 11$ $\beta \neq 0$	8.4
34	$1, 2, 3; 11$	10.1
35	$2, 3; \alpha 1 + 5; 4 + \beta 6 + 10$	6.23
36	$2, 3; \alpha 1 + 5; 6 + 10$	6.22
37	$2, 3; 1 + 5; 10$	6.22
38	$2, 3; 5; 10$	7.12°
39	$1, 2, 3; 4 + 10$	8.3°

i	Basis	Nor
	$r = 4$	
40	$1, 2, 3; 10$	11,1
41	$1; \sigma 2 + \tau 3 + 4, \alpha 3 + 5, \beta 2 + 6$ $\sigma^2 + \tau^2 + (\alpha + \beta)^2 = 1$	6,24
42	$1; 4, 3 + 5, 2 - 6$	7,8°
43	$1; 4, 5, 6$	8.4
44	$2, \alpha 1 + 3; 1 + 5, 6$ $\alpha \neq 0$	6,24
45	$2, 3; 1 + 5, 6$	7,14
46	$2, \alpha 1 + 3; 5, 6$ $\alpha \neq 0$	7,13
47	$2, 3; 5, 6$	9,1
48	$1, 2; 3 + 5, 6$	6.24
49	$1, 2; 5, 6$	7,13
50	$1, 2, 3; 4$	9,1
	$r = 3$	
1	$7; 10; 11$	= 3,1
2	$1; 10; \beta 4 + 7 + \alpha 11$	5,2
3	$4; 7; 11$	= 3,3
4	$1; \alpha 4 + 7; \beta 4 + 11$	4.6
5	$5, 6; \beta 4 + 7 + \alpha 11$ $\alpha \neq 0$	5,5
6	$1; 4; 7 + \alpha 11$ $\alpha \neq 0$	4,6
7	$2, 3; \beta 4 + 7 + \alpha 11$ $\alpha \neq 0$	5,6
8	$7, 8, 9$	5,1
9	$1; \alpha 4 + 7; 4 + 10$	4,4°
10	$5, 6; \beta 4 + 7$	6,4
11	$1; 4; 7$	5.2
12	$2, 3; \beta 4 + 7$ $\beta \neq 0$	6,5
13	$2, 3; 7$	7.4
14	$5, 6; 1 + \alpha 4 + 7$	5,9°
15	$3 + 5, 2 - 6; \alpha 1 + \beta 4 + 7$	5:16
16	$2, 3; 1 + 7$	6,3°
17	$1; 4; 7 + 10$	4,4°
18	$2, 3; \beta 4 + 7 + \beta 10$ $\beta \neq 0$	5.12°
19	$2, 3; 7 + 10$	5,4°°
20	$1; 10; \beta 4 + 11$	5,2
21	$5, 6; \beta 4 + 11$	5,5
22	$1; \alpha 4 + 6; \beta 5 + \sigma 5 + 11$	5.24
23	$1; 4; \sigma 6 + 11$ $\sigma \neq 0$	5,24
24	$1; 4; 11$	6,4
25	$2, 3; \beta 4 + \sigma 5 + 11$ $\sigma \neq 0$	6,21
26	$2, 3; \beta 4 + 11$	7,6
27	$3; \alpha 1 + \beta 2 + 6; 4 + 10$	5,34
28	$1; 2 + 4; 10$	5,22°
29	$1; 4; 10$	7,4

Table 6 (*Continued*)

i	Basis	Nor
	$r = 3$	
30	$2, 3; 4 + \sigma 6 + 10$	6,23
	$\sigma \neq 0$	
31	$2, 3; 4 + 10$	7,10°
32	$2, 3; 6 + 10$	6,22
33	$2, 3; 10$	8,2°°
34	$-\delta 2 + \beta 3 + 4, \delta 1 + \sigma 2 - \alpha 3 + 5,$	6,24
	$-\beta 1 + \alpha 2 + \tau 3 + 6$	
	$\alpha^2 + \beta^2 + \delta^2 + (\sigma + \tau)^2 = 1$	
35	$4,3 + 5,2 - 6$	7,8°
36	$1 + 4, 5, 6$	7,8°
37	$4, 5, 6$	10,1
38	$\alpha 1 + 3; \beta 1 + 5, \sigma 1 + \tau 2 + 6$	6,24
	$\beta^2 + \sigma^2 + \tau^2 = 1$	
39	$\alpha 1 + 3; 5, 6$	7,13
40	$1, 3 + 5, \tau 2 + 6$	6,24
	$\tau \neq -1$	
41	$1, 3 + 5, 2 - 6$	7,8°
42	$1, 5, 6$	8,4
43	$\beta 1 + 3, 2; 4$	7,13
44	$2, 3; 4$	8,4
45	$1, 2; 3 + 4$	7,14
46	$1, 2; 4$	8,5
47	$1, 2, 3$	11,1
	$r = 2$	
1	$10; 7 + \alpha 11$	3,1
	$\alpha \neq 0$	
2	$\alpha 4 + 7; \beta 4 + 11$	3,3
3	$4; 7 + \alpha 11$	3,3
	$\alpha \neq 0$	
4	$1; \beta 4 + 7 + \alpha 11$	4,6
	$\alpha \neq 0$	
5	$7; 10$	4,2°
6	$1 + 7; 10$	3,2°°
7	$\alpha 1 + 7; 4 + 10$	3,9°
8	$4; 7$	4,6
9	$1; \beta 4 + 7$	5,2
10	$4; 1 + 7$	3,11
11	$1; \beta 4 + 7 + 10$	4,4°

i	Basis	Nor
	$r = 2$	
12	$10; 11$	5,1
13	$4; 11$	5,5
14	$4; \alpha 5 + 11$	4,26
	$\alpha \neq 0$	
15	$1; \beta 4 + \alpha 5 + 11$	5,24
	$\alpha \neq 0$	
16	$1; \beta 4 + 11$	6,4
17	$1; 10$	7,4
18	$3; 4 + \alpha 6 + 10$	5,34
	$\alpha \neq 0$	
19	$1; 4 + 10$	6,3°
20	$\alpha 1 + \sigma 3 + 5, \beta 1 + \tau 2 + 6$	6,24
	$\alpha^2 + \beta^2 + (\sigma + \tau)^2 = 1$	
21	$3 + 5, 2 - 6$	7,8°
22	$5, 6$	8,4
23	$\alpha 1 + 2; 3 + 4$	6,24
24	$\alpha 1 + 2; 4$	7,13
25	$1; 3 + 4$	7.14
26	$1; 4$	9,1
27	$2, 3$	9,1
	$r = 1$	
1	$\beta 4 + 7 + \alpha 11$	3,3
	$\alpha \neq 0$	
2	$\beta 4 + 7$	4,6
	$\beta \neq 0$	
3	7	5,2
4	$1 + 7$	4,4°
5	$\beta 4 + 7 + \beta 10$	3,9°
	$\beta \neq 0$	
6	$7 + 10$	3,2°°
7	$\beta 4 + 11$	5,5
	$\beta \neq 0$	
8	11	7,2
9	$4 + 10$	5,12°
10	10	8,1
11	$3 + 4$	6,24
12	4	8,4
13	1	9,1

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Translated by E.L.S.